Matrices

An $m \times n$ matrix is a collection of real numbers organized in m rows and n columns

Ex: 2×3 matrix $\begin{bmatrix} -1 & 5 & \frac{2}{3} \\ 0 & 1 & 6 \end{bmatrix}$

Ex: An $n \times 1$ matrix is an n-dimensional vector

Addition: Let A and B be $m \times n$ matrices. Then $A+B$ is the $m \times n$ matrix formed by adding the corresponding entries of A and B

Ex: $\begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & \frac{1}{2} \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 8 \end{bmatrix}$

Scalar Multiplication: Let A be $m \times n$ and $c \in \mathbb{R}$. Then cA is the $m \times n$ matrix obtained by multiplying every entry of A by c.

Ex: $3 \cdot \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 3 \\ 6 & 3 & 9 \end{bmatrix}$

Matrix-Vector Multiplication

Let A be an $m \times n$ matrix and $v \in \mathbb{R}^n$

Then Av is a vector in $\mathbb{R}^m$ given by the linear combination of the columns of A with weights in v.

Ex: $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$

Alternatively, we can compute Av by performing the dot product of each row of A with v:

Ex: $\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(1) + 1(2) \\ 1(1) + 0(2) \\ 0(1) + -2(2) \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$

Matrix Multiplication

Let A be $m \times n$ and B be $n \times p$.

Then AB is the $m \times p$ matrix obtained by multiplying each column of B by A.

Ex: $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 5 & 1 & 0 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} A\begin{bmatrix} -3 \\ 1 \\ 5 \end{bmatrix} & A\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 1 & -1 & 1 \\ 4 & 2 & 1 & 2 \end{bmatrix}$$

A quicker way to compute:

The (i,j)-th entry, denoted A_{ij}, of AB is the dot product of the i^{th} row of A with the j^{th} column of B

Ex: $A = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 5 & 1 & 0 & 0 \end{bmatrix}$

$(1,1)$-entry of AB is $(-1)(-3)+(0)(1)+(1)(5) = 8$

Continuing in this way,

$$AB = \begin{bmatrix} 8 & 1 & -1 & 0 \\ 5 & 2 & 2 & 1 \end{bmatrix}$$

Note: BA is not defined

Properties:

- $A+B = B+A$
- $A+(B+C) = (A+B)+C$
- $c(A+B) = cA+cB$
- $c(AB) = (cA)(B) = A(cB)$
- $A(B+C) = AB+AC$
- $(B+C)A = BA+CA$

But: $AB \neq BA$ in general

Ex: $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$ but $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$

A special matrix

The $n \times n$ identity matrix I_n has 1's on the diagonal and 0's elsewhere

Ex: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

I_n has the property that $I_n \cdot v = v \quad \forall \; v \in \mathbb{R}^n$

and $A \cdot I_n = A \quad \forall \; n \times n$ matrices A.

Question: Why are matrix operations defined the way they are?

In calculus, you deal with functions $f: \mathbb{R} \to \mathbb{R}$ (1-dimensional vectors)

E.g. $f(x) = x^2$, $g(x) = \sin x$

We will consider functions $f: \mathbb{R}^n \to \mathbb{R}^m$

Ex: $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x^2 \\ y \\ 2x-y \end{bmatrix}$

$f\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$